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Markov inequality for polynomials of degree n with m distinct zeros

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Abstract

Let $\mathscr{P}_n^m$ be the collection of all polynomials of degree at most n with real coefficients that have at most m distinct complex zeros. We prove that

$$\max_{x\in[0,1]} |P'(x)| \leqslant 32\cdot 8^m n \max_{x\in[0,1]} |P(x)|$$

for every $P \in \mathscr{P}_n^m$. This is far away from what we expect. We conjecture that the Markov factor $32\cdot 8^m n$ above may be replaced by cmn with an absolute constant $c>0$. We are not able to prove this conjecture at the moment. However, we think that our result above gives the best-known Markov-type inequality for $\mathscr{P}_n^m$ on a finite interval when $m \leqslant c\log n$.

Keywords: Markov-type inequalities; Polynomials with restricted zeros

1. Introduction, notation, new result

Markov's inequality asserts that

$$\max_{x\in[0,1]} |P'(x)| \leqslant 2n^2 \max_{x\in[0,1]} |P(x)|$$

for all polynomials of degree at most n with real coefficients. There is a huge literature about Markov-type inequalities for constrained polynomials. In particular,

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several essentially sharp improvements are known for various classes of polynomials with restricted zeros. Here we just refer to [1], and the references therein.

Let $\mathcal{P}_n^m$ be the collection of all polynomials of degree at most n with real coefficients that have at most m distinct complex zeros. We prove the following.

Theorem. *We have*

$$\max_{x\in[0,1]} |P'(x)| \leqslant 32\cdot 8^m n \max_{x\in[0,1]} |P(x)|$$

for every $P\in\mathcal{P}_n^m$.

This is far away from what we expect. We conjecture that the Markov factor $32\cdot 8^m n$ above may be replaced by cmn with an absolute constant $c>0$. We are not able to prove this conjecture at the moment. However, we think that our result above gives the best-known Markov-type inequality for $\mathcal{P}_n^m$ on a finite interval when $m \leqslant c\log n$.

2. Proof

It is easy to see by Rouche's Theorem that $\mathcal{P}_n^m$ is closed in the maximum norm on $[0,1]$, and hence in any norm. Therefore, it is easy to argue that there is a $P^*\in\mathcal{P}_n^m$ with minimal L_1 norm on $[0,1]$ such that

$$\frac{|P^{*'}(0)|}{\max_{x\in[0,1]} |P^*(x)|} = \sup_{P\in\mathcal{P}_n^m} \frac{|P'(0)|}{\max_{x\in[0,1]} |P(x)|}.$$

Lemma 1. *There is a polynomial* $T\in\mathcal{P}_n^{m+1}$ *of the form*

$$T(x) = Q(x)(x-a),$$

where $Q\in\mathcal{P}_{n-1}^m$ *has all its zeros in* $[0,1]$, $a\in\mathbb{R}$, *and*

$$\frac{|P^{*'}(0)|}{\max_{x\in[0,1]} |P^*(x)|} \leqslant \frac{|T'(0)|}{\max_{x\in[0,1]} |T(x)|}.$$

Proof. Assume that $z_0\in\mathbb{C}\backslash\mathbb{R}$ is a zero of P^* with multiplicity k. Then

$$P_\varepsilon^*(x) := P^*(x)\left(1-\varepsilon\frac{x^2}{(x-z_0)(x-\bar{z}_0)}\right)^k$$

with a sufficiently small $\varepsilon>0$ is in $\mathcal{P}_n^m$ and it contradicts the defining properties of P^*. So each of the zeros of P^* is real. Now let $P^* = RS$ where all the zeros of R are in $[0,1]$, while $S(0)>0$ and all the zeros of S are in $\mathbb{R}\backslash[0,1]$. We may assume that S is not identically constant, otherwise $T := P^*\in\mathcal{P}_n^{m+1}$ with $Q\in\mathcal{P}_{n-1}^m$

defined by

$$Q(x) := \frac{P^*(x)}{x-a}$$

is a suitable choice, where $x-a$ is any linear factor of P^*. It is easy to see that S can be written as

$$S(x) := \sum_{j=0}^{d} A_j x^j (1-x)^{d-j}, \quad A_j \geqslant 0,\ j=0,1,\dots,d,$$

where $d \geqslant 1$ is the degree of S. Now let

$$T(x) = R(x) \sum_{j=0}^{1} A_j x^j (1-x)^{d-j}.$$

Then T is of the form

$$T(x) = Q(x)(x-a),$$

where $Q \in \mathscr{P}_{n-1}^m$ has all its zeros in $[0,1]$, $a \in \mathbb{R}$, and

$$\frac{|P^{*'}(0)|}{\max_{x\in[0,1]} |P^*(x)|} \leqslant \frac{|T'(0)|}{\max\limits_{x\in[0,1]} |T(x)|},$$

and the proof is finished. □

For the sake of brevity let

$$n \leqslant M(n,m) := \sup_P \frac{|P'(0)|}{\max_{x\in[0,1]} |P(x)|},$$

where the supremum is taken for all $P \in \mathscr{P}_n^m$ having all their zeros in $[0,1]$.

Lemma 2. *Let P^* and $T(x) = Q(x)(x-a)$ be as in Lemma* 1. *Suppose $a<0$ or $a>2$. Then*

$$\max_{x\in[0,1]} |Q(x)| \leqslant 4M(n,m) \max_{x\in[0,1]} |T(x)|.$$

Proof. Let $b \in [0,1]$ be a point for which

$$|Q(b)| = \max_{x\in[0,1]} |Q(x)|.$$

Case 1: $b \in [1/2, 1]$. In this case

$$\max_{x\in[0,1]} |Q(x)| = |Q(b)| = \frac{|T(b)|}{|b-a|} \leqslant 2|T(b)| \leqslant 2 \max_{x\in[0,1]} |T(x)|.$$

Case 2: $b \in [0, 1/2]$. In this case $Q = UV$, where $U \in \mathcal{P}_n^m$ has all its zeros in $[b, 1]$, and $V \in \mathcal{P}_n^m$ has all its zeros in $\mathbb{R}\setminus[b, 1]$. It is easy to see that V can be written as

$$V(x) := \sum_{j=0}^{d} B_j (x-b)^j (1-x)^{d-j}, \quad B_j \geqslant 0, \quad j = 0, 1, \dots, d,$$

where d is the degree of V. Now let

$$W(x) = U(x) B_0 (1-x)^d.$$

Then

$$|W(b)| = |(UV)(b)| = |Q(b)| = \max_{x \in [b,1]} |Q(x)| \tag{1}$$

and

$$|W(x)| \leqslant |Q(x)|, \quad x \in [b, 1]. \tag{2}$$

Also $W \in \mathcal{P}_n^m$ has all its zeros in $[b, 1]$. Let $\eta > b$ be the smallest point for which

$$|W(\eta)| = \tfrac{1}{2} \max_{x \in [b,1]} |W(x)|.$$

Then $|W'(x)|$ is decreasing on $[b, \eta]$, and it follows by a linear transformation that

$$|W'(b)| \leqslant \frac{M(n,m)}{1-b} \max_{x \in [b,1]} |W(x)| \leqslant 2M(n,m) \max_{x \in [b,1]} |W(x)|. \tag{3}$$

Combining the above by the mean value theorem, we obtain

$$\begin{aligned} \tfrac{1}{2} \max_{x \in [b,1]} |W(x)| &= |W(b) - W(\eta)| = (\eta - b)|W'(\xi)| \\ &\leqslant (\eta - b)|W'(b)| \leqslant (\eta - b) 2M(n,m) \max_{x \in [b,1]} |W(x)|, \end{aligned}$$

whence

$$(\eta - b) \geqslant (4M(n,m))^{-1}.$$

This, together with (1)–(3), yields

$$\max_{x \in [0,1]} |Q(x)| \leqslant 2|Q(\eta)| = \frac{|T(\eta)|}{\eta - a} \leqslant 4M(n,m) \max_{x \in [0,1]} |T(x)|,$$

and the proof is finished. $\square$

Lemma 3. *Let P^* be as in Lemma* 1. *Then there exists a polynomial $U \in \mathcal{P}_n^{m+1}$ having all its zeros in* $[0, 1]$ *such that*

$$\frac{|U'(0)|}{\max_{x \in [0,1]} |U(x)|} \geqslant \frac{1}{7} \frac{|P^{*\prime}(0)|}{\max_{x \in [0,1]} |P^*(x)|}.$$

Proof. Let $T(x) = Q(x)(x - a)$ as in Lemma 1. We distinguish three cases.

Case 1: $a \in [0, 1]$. In this case $U(x) = T(x)$ is a suitable choice.

Case 2: $a \in [1, 2]$. In this case $U(x) = T(ax)$ is a suitable choice.

Case 3: $a<0$ or $a>2$. Then we have

$$T'(0) = -aQ'(0) + Q(0).$$

Combining this with Lemma 2 we obtain

$$\begin{aligned}\frac{|P^{*'}(0)|}{\max_{x\in[0,1]}|P^*(x)|} &\leqslant \frac{|T'(0)|}{\max\limits_{x\in[0,1]}|T(x)|} \leqslant \frac{|aQ'(0)|}{\max\limits_{x\in[0,1]}|Q(x)(x-a)|} \\ &\quad + \frac{|Q(0)|}{\max\limits_{x\in[0,1]}|Q(x)(x-a)|} \\ &\leqslant \frac{|aQ'(0)|}{|\frac{a}{2}|\max\limits_{x\in[0,1]}|Q(x)|} + \frac{|Q(0)|}{(4M(n,m))^{-1}\max\limits_{x\in[0,1]}|Q(x)|} \\ &\leqslant 2M(n-1,m) + 4M(n,m+1) \leqslant 6M(n,m).\end{aligned}$$

This means that there is a polynomial $U\in\mathscr{P}_n^{m+1}$ having all its zeros in $[0,1]$ such that

$$\frac{|U'(0)|}{\max_{x\in[0,1]}|U(x)|} \geqslant (1/7)\frac{|P^{*'}(0)|}{\max\limits_{x\in[0,1]}|P^*(x)|}. \qquad \square$$

We introduce

$$n \leqslant M^*(n,m) := \sup_P \frac{|P'(0)|}{\max_{x\in[0,1]}|P(x)|},$$

where the supremum is taken for all $P\in\mathscr{P}_n^m$ having all their zeros in $[0,1]$ for which

$$|P(0)| = \max_{x\in[0,1]}|P(x)|.$$

Lemma 4. *We have* $M(n,m+1) = M^*(n,m+1)$.

Proof. Since $M(n,m+1) \geqslant M^*(n,m+1)$ is trivial, we need to see only $M(n,m+1) \leqslant M^*(n,m+1)$. To this end take a $P\in\mathscr{P}_n^{m+1}$ and choose $\alpha\in(-\infty,0]$ so that

$$|P(\alpha)| = \max_{x\in[0,1]}|P(x)|.$$

Now let

$$U(x) := P((1-\alpha)x+\alpha).$$

Then $U\in\mathscr{P}_n^{m+1}$ has all its zeros in $[0,1]$ and

$$|U(0)| = |P(\alpha)| = \max_{x\in[0,1]}|P(x)| = \max_{x\in[\alpha,1]}|P(x)| = \max_{x\in[0,1]}|U(x)|,$$

while, since $|P'(x)|$ is decreasing on $(-\infty,0]$, we have

$$|U'(0)| = (1-\alpha)|P'(\alpha)| \geqslant (1-\alpha)|P'(0)| \geqslant |P'(0)|.$$

Therefore

$$\frac{|P'(0)|}{\max_{x\in[0,1]} |P(x)|} \leqslant \frac{|U'(0)|}{\max\limits_{x\in[0,1]} |U(x)|}. \qquad \square$$

From Lemmas 3 and 4 we can draw the following conclusion.

Lemma 5. *We have*

$$\sup_{P\in\mathscr{P}_n^m} \frac{|P'(0)|}{\max_{x\in[0,1]} |P(x)|} \leqslant 7M^*(n, m+1).$$

Lemma 6. *We have* $M^*(n,m) \leqslant \frac{2}{7} 8^m n$.

Proof. Suppose that $P \in \mathscr{P}_n^m$ has all its zeros in $[0,1]$, and

$$|P(0)| = \max_{x\in[0,1]} |P(x)|.$$

Let $F(x) := |P(x)|^{1/d}$, where $d(\leqslant n)$ is the degree of P. Then

$$|F(0)| = \max_{x\in[0,1]} |F(x)|. \tag{4}$$

Let

$$F(x) = \prod_{i=1}^{m} |x - x_i|^{\alpha_i},$$

where

$$0 < x_1 < \cdots < x_m < 1, \quad 0 < \alpha_i, \quad i = 1,2,\ldots,m, \quad \sum_{i=1}^{m} \alpha_i = 1.$$

We show that

$$\frac{\alpha_i}{x_i} \leqslant 2 \cdot 8^{m-i} \tag{5}$$

for each $i = 1,2,\ldots,m$. To see this let

$$A_1 := \{1,2,\ldots,i_1\},$$

$$A_2 := \{i_1+1, i_1+2, \ldots, i_2\},$$

$$\vdots$$

$$A_\mu := \{i_{\mu-1}+1, i_{\mu-1}+2, \ldots, i_\mu := m\},$$

be the sets of indices for which

$$\frac{x_{i+1}}{x_i} \leqslant 8 \quad \text{whenever } i \text{ and } i+1 \text{ are in the same } A_\nu,$$

$$\frac{x_{i+1}}{x_i} > 8 \quad \text{whenever } i \text{ and } i+1 \text{ are not in the same } A_\nu.$$

Inequality (5) is clear for any $i \in A_\mu$, since (4) implies that

$$\frac{\alpha_i}{x_i} \leqslant \frac{1}{x_i} \leqslant \frac{8^{m-i}}{x_m} \leqslant 2 \cdot 8^{m-i}.$$

We continue by induction. Assume that (5) holds for any $i \in A_\nu \cup A_{\nu+1} \cup \cdots \cup A_\mu$. We prove that it holds for any $j \in A_{\nu-1}$. Since

$$\prod_{i=1}^{m} |x - x_i|^{\alpha_i} \leqslant F(0) = \prod_{i=1}^{m} |x_i|^{\alpha_i}, \quad x \in [0,1],$$

we have

$$\sum_{i=1}^{m} \alpha_i \log\left|\frac{x}{x_i} - 1\right| \leqslant 0, \quad x \in [0,1].$$

Let $j \in A_{\nu-1}$ be arbitrary and $x^* := 4x_{i_{\nu-1}}$. For $k \in A_\nu \cup A_{\nu+1} \cup \cdots \cup A_\mu$ we have $x^*/x_k \leqslant 1/2$, so

$$\log\left(1 - \frac{x^*}{x_k}\right) \geqslant -2(\log 2) \cdot \frac{x^*}{x_k}.$$

Thus

$$(\log 3) \sum_{i=1}^{i_{\nu-1}} \alpha_i \leqslant 2(\log 2) \cdot x^* \sum_{i=i_{\nu-1}+1}^{m} \frac{\alpha_i}{x_i},$$

$$\frac{\alpha_j}{x_j} \leqslant \frac{2(\log 2)}{\log 3} \frac{x^*}{x_j} \sum_{i=i_{\nu-1}+1}^{m} \frac{\alpha_i}{x_i} \leqslant \frac{2(\log 2)}{\log 3} 4 \cdot 8^{i_{\nu-1}-j} (2 + 2 \cdot 8 + \cdots + 2 \cdot 8^{m-i_{\nu-1}-1}),$$

from which

$$\frac{\alpha_j}{x_j} \leqslant 2 \cdot 8^{m-j}$$

follows immediately. The proof of (5) is now complete for all $i = 1, 2, \ldots, m$. The lemma follows by:

$$\frac{|P'(0)|}{|P(0)|} = d \frac{|F'(0)|}{|F(0)|} \leqslant d \frac{2}{7} 8^m. \qquad \square$$

The following is a consequence of Lemmas 5 and 6.

Corollary 7. *We have*

$$|P'(0)| \leqslant 2 \cdot 8^{m+1} n \max_{x \in [0,1]} |P(x)|.$$

for every $P \in \mathscr{P}_n^m$.

Proof of the Theorem. We need to prove that

$$|P'(y)| \leqslant 4 \cdot 8^{m+1} n \max_{x \in [0,1]} |P(x)|$$

for every $P \in \mathscr{P}_n^m$ and $y \in [0,1]$. However, it follows from Corollary 7 by a simple linear transformation that

$$|P'(y)| \leqslant 2 \cdot 2 \cdot 8^{m+1} n \max_{x \in [y,1]} |P(x)| \leqslant 4 \cdot 8^{m+1} n \max_{x \in [0,1]} |P(x)|, \quad y \in [0, 1/2],$$

and

$$|P'(y)| \leqslant 2 \cdot 2 \cdot 8^{m+1} n \max_{x \in [0,y]} |P(x)| \leqslant 4 \cdot 8^{m+1} n \max_{x \in [0,1]} |P(x)|, \quad y \in [1/2, 1].$$

This finishes the proof. □

References

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